<Lecture Note>
Process Dynamics and Control

- Digital Control -

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Part Six : Digital Control Technique

Chap 21. Digital Computer Control

Microprocessor
Analog Controller -------------> Digital Computer Control

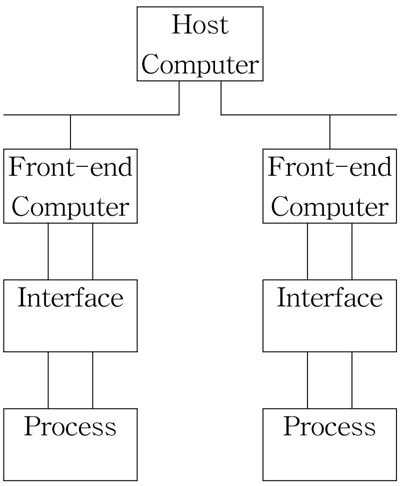
Economically justifiable nowadays!
⇒ **Choice of correct system – single/multiple computer, microprocessor based system, ...**

* Role for digital computer system in Process Control

Passive application : Data acquisition (manipulation of process data)

Active application : Manipulation of the process by computer

<Distributed data processing>



Complex calculation
Supervisory control

Data acquisition

A/D, D/A, Transmitter,....amplifier...
4-20mA, 1-5V, 3-15psig

Process + sensor & actuator

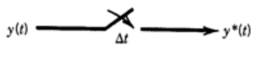
Sensor	Transmitter	ADC				DAC	Transmitter	Actuator
0-100℃	4-20mA	12bit	→		→	4-20mA	mA	Valve opening
5-200psig	1-5V	(0-4095)	→	Computer	→	1-5V	1-5V	Inverter output
0-50gpm	0-10V	16bit, . . .	→		→	0-10V	0-10V	Displacement
. . .	0.04mV/℃					+ Hold		

Chap 22. Sampling and Filtering of continuous Measurement

For data acquisition and control with computer based system, Specify

1. Sampling rate
2. S/N ratio, signal conditioning
3. control law

22.1. Sampling and Signal Reconstruction



(a) The sampler

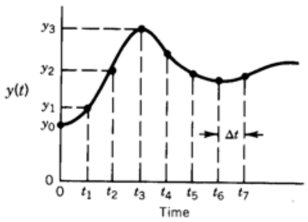
Sampling Period : Δt (min)

Sampling Rate : $f_s = \dfrac{1}{\Delta t}$ (cycle / min)

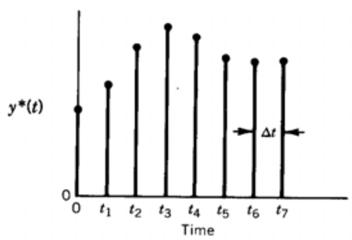
Sampling freq. : $w_s = \dfrac{2\pi}{\Delta t}$ (rad / min)

Continuous Signal Sampled Signal

$y(t)$ ----------→ $y^*(t)$
Sampler



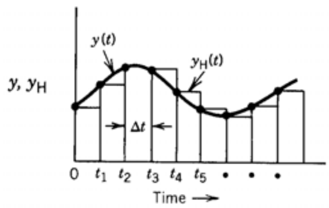
(b) Continuous signal y(t) and samples



(c) Sampled signal

* Impulse modulation : representation of sampled signal

digital signal ------------→ continuous signal
Signal Reconstruction
by DAC + Hold



(b) Comparison of original signal, y(t), and reconstructed signal, $y_H(t)$

Zero-order hold : $\bar{y}_H = \bar{y}_{n-1}$ for $t_{n-1} \le t < t_n$

$$H_o(s) = \frac{1-e^{-s\Delta t}}{s}$$

<disadvantages of ZOH>

1. ZOH starts attenuating at freq.'s considerably below w_s
2. ZOH allows high freq.'s to pass although they are attenuated.
3. ZOH provides apparent time delay of one half of sampling period

$$H(s) = \frac{1-e^{-s\Delta t}}{s} \quad \underset{=>}{(s \to j\omega)} \quad H(j\omega) = \frac{2e^{-j\omega\Delta t/2}(e^{j\omega\Delta t/2} - e^{-j\omega\Delta t/2})}{2j\omega} = \frac{\Delta t \sin(\omega\Delta t/2)}{\omega\Delta t/2} e^{-j\omega\Delta t/2}$$

$$HG^*(j\omega) = \frac{1}{\Delta t}\sum_{n=-\infty}^{\infty} H(j\omega + jn\omega_s)\, G(j\omega + jn\omega_s) \cong e^{-s\Delta t/2} G(s)$$

First-order hold: $\bar{y}_H = \bar{y}_{n-1} + \left(\frac{t-t_{n-1}}{\Delta t}\right)(\bar{y}_{n-1} - \bar{y}_{n-2})$ for $t_{n-1} \le t < t_n$

$$H_1(s) = \frac{1+\Delta t s}{\Delta t}\left(\frac{1-e^{-s\Delta t}}{s}\right)^2$$

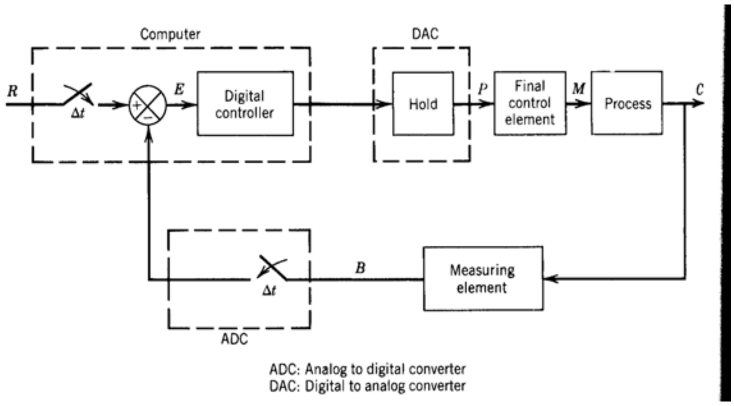
→ High order holds do not offer significant advantages for most control problem
→ ZOH is most widely used.

Fractional order Hold: $\bar{y}_H = \bar{y}_{n-1} + \alpha\left(\frac{t-t_{n-1}}{\Delta t}\right)(\bar{y}_{n-1} - \bar{y}_{n-2})$ for $t_{n-1} \le t < t_n$

Continuous Slewing: $\bar{y}_H = \bar{y}_{n-2} + \left(\frac{t-t_{n-1}}{\Delta t}\right)(\bar{y}_{n-1} - \bar{y}_{n-2})$ for $t_{n-1} \le t < t_n$

$$H_c(s) = \frac{1}{\Delta t}\left(\frac{1-e^{-s\Delta t}}{s}\right)^2$$

<Block diagram of digital control system>



* Multirate sampling : use of different sampling period in the same system

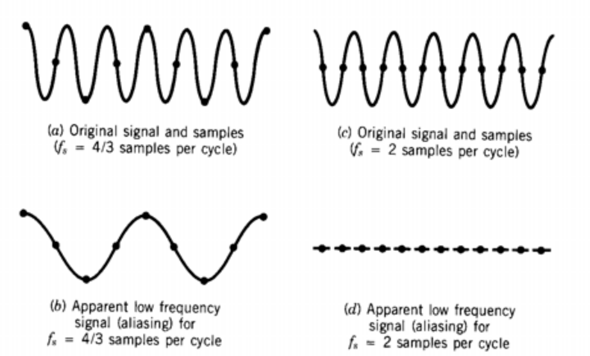
22.2. Selection of the Sampling Period

Selection of Δt based on

1. # of measurement
2. view point of process control

<aliasing> --- foldover effect

Sampling rate must be large enough not to loose significant process information



* Shannon's Sampling theorem :
 - Sampling freq. must be more than twice the freq. of sine wave(original signal)

"To be able to recover the continuous signal from its sampled counterpart, the sampling freq. (ω_s) must be at least twice the highest freq. (ω_c) in the signal."

⇒ actual signal has all freq.
⇒ it is impossible to recover the continuous signal from its sampled counterpart.

Aliasing can happen for the sampling of nonsinusoidal signal.
⇒ Use anti-aliasing filter

<Selection of Sampling Period>

* S/N ratio = $\frac{\sigma_s^2}{\sigma_N^2}$ (σ^2 : variance)

based on process variable type : Flow, level, temp...
based on open-loop system : τ, Θ, τmax, settling time, ωc
⇒ Selection of sampling period is less significant nowadays!

22.3. Signal Processing and Data Filtering

Noise Sources : measurement device → filtering
electrical equipment → Shielding, grounding
process itself → filtering
(multiphase flow, mixing, turbulence, . . .)
filter = Transfer function !

<Analog filter>

$$\tau_F \frac{dy(t)}{dt} + y(t) = x(t)$$

τ_F : Filter time constant, $y(t)$: filtered value, $x(t)$: measured value

- steady-state gain of the filter = 1
- exponential filter = Low pass filter = RC filter
- Before Sampling, use analog filter to reduced the noise
  → prefiltering → anti-aliasing filter

- if $\tau_F < 3$ sec, use passive analog filter (R, C....)
  if $\tau_F > 3$ sec, use active analog filter (op amplifier)

- Usually $\tau_F < 0.1 \tau_{max}$
- τ_F should be selected so that (lowest noise freq. w_N)

$$w_N > w_F = \frac{2\pi}{\tau_F} > w_{max} = \frac{2\pi}{\tau_{max}}$$

<Digital filters>

* Exponential filter
  x_{n-1}, x_n..... : measured
  y_{n-1}, y_n ..... : filtered

Use of backward difference approximation for analog filter

$$\tau_F \frac{y_n - y_{n-1}}{\Delta t} + y_n = x_n$$

$$y_n = \frac{\Delta t}{\tau_F + \Delta t} x_n + \frac{\tau_F}{\tau_F + \Delta t} y_{n-1}$$

$\alpha = \dfrac{1}{\tau_F / \Delta t + 1}$: filter constant $\Rightarrow$ $\tau_F = \dfrac{\Delta t(1-\alpha)}{\alpha}$

$y_n = \alpha x_n + (1-\alpha) y_{n-1}$: Weighted summation!

$\Rightarrow$ **single exponential smoothing**

$\alpha = 1$: No filtering ($\tau_F = 0$), $\alpha = 0$: measurement is ignored ($\tau_F = \infty$),

<Double Exponential Filter>

----> [Exp. filter] ----> [Exp. filter] ---->

x_n y_n $\bar{y}_n$

$$\bar{y}_n = \gamma y_n + (1-\gamma)\, \bar{y}_{n-1}$$

$$\bar{y}_n = \gamma \alpha x_n + \gamma(1-\alpha)\, y_{n-1} + (1-\gamma)\, \bar{y}_{n-1}$$

($\bar{y}_{n-1} = \gamma y_{n-1} + (1-\gamma)\, \bar{y}_{n-2}$ => $y_{n-1} = \frac{1}{\gamma}\, \bar{y}_{n-1} - \frac{1-\gamma}{\gamma}\, \bar{y}_{n-2}$)

$$= \gamma \alpha x_n + (2-\gamma-\alpha)\, \bar{y}_{n-1} - (1-\alpha)(1-\gamma)\, \bar{y}_{n-2}$$

Simplification : $\gamma = \alpha$

$$\bar{y}_n = \alpha^2 x_n + 2(1-\alpha)\, \bar{y}_{n-1} - (1-\alpha)^2\, \bar{y}_{n-2}$$

$\Rightarrow$ better filtering for high freq. noise than exponential filter

<Moving average filter>

$y_n = \dfrac{1}{J} \sum_{i=n-J+1}^{n} x_i$ J: moving window size

$$y_{n-1} = \frac{1}{J} \sum_{i=n-J}^{n-1} x_i$$

$y_n = y_{n-1} + \dfrac{1}{J} (x_n - x_{n-J})$ (recursive form) : low pass filter

<Noise spike filter (rate of change filter)>

$$\bar{y}_n = \begin{cases} x_n & \text{if } \quad |x_n - y_{n-1}| \le \Delta x \\ y_{n-1} - \Delta x & \text{if } \quad y_{n-1} - x_n > \Delta x \\ y_{n-1} + \Delta x & \text{if } \quad y_{n-1} - x_n < -\Delta x \end{cases} \qquad (\Delta x \text{ : maximum change rate})$$

⇒ **use for power glitch, instrument glitches, . . .**

22.4. Comparison of Analog and Digital Filter

1. Digital filters can be easily tuned (programmed) to fit the process. They are also easily modified.
2. Digital filters require the choice of sampling period
3. Digital filters affect the performance of computer system
4. Analog filters are particularly effective for elimination of high-frequency noise and aliasing

22.5. Effect of filter selection on control system performance

filter → dynamic element → phase lag added → reduced stability limit

filter constant change → retune the controller!

To compensate the lag → D-mode in PID

Chap 23. Development of Discrete-Time Models

Discretize
Continuous-time model ------------> Discrete-time model
O.D.E difference eqn.

23.1 Finite Difference Model

$\dfrac{dy(t)}{dt} = f(y, x)$: y = output, x = input

by finite difference approximation (backward difference),

$$\frac{y_n - y_{n-1}}{\Delta t} \cong f(y_{n-1}, x_{n-1})$$: Calculated from known quantities

$\Rightarrow$ $y_n = y_{n-1} + \Delta t\, f(y_{n-1}, x_{n-1})$: Recurrence relation

* It can be used for numerical integration (Euler integration)

Use of forward difference

$$\frac{dy}{dt} \cong \frac{y_{n+1} - y_n}{\Delta t} \qquad [\textit{shifting index} : n \rightarrow (n-1)]$$

Interpolation formula : $\dfrac{dy}{dt} = \dfrac{y(k) + 3y(k-1) - 3y(k-2) - y(k-3)}{6\Delta t}$

23.2 Exact Discretization of Linear Systems

For Linear D.E. and piecewise constant input (during the sampling period)

$$\tau \frac{dy(t)}{dt} + y(t) = x(t) \; ; \quad x(t) = x(0) \; (constant), \quad y(0) \neq 0$$

Laplace Transform : $sY(s) - y(0) = \dfrac{-1}{\tau} Y(s) + \dfrac{1}{\tau}\dfrac{x(0)}{s}$

$$Y(s) = \frac{1}{s + 1/\tau}\left[\frac{1}{\tau}\frac{x(0)}{s} + y(0)\right]$$

Inverse L. T. : $y(t) = x(0)(1 - e^{-t/\tau}) + y(0)\, e^{-t/\tau}$

$$\therefore \; y(\Delta t) = x(0)(1 - e^{-\Delta t/\tau}) + y(0)\, e^{-\Delta t/\tau}$$

$$\vdots$$

$$y(n\Delta t) = \underbrace{x((n-1)\Delta t)(1 - e^{-\Delta t/\tau})}_{\text{input (constant)}} + \underbrace{y((n-1)\Delta t) e^{-\Delta t/\tau}}_{\text{initial condition}}$$

$$\Downarrow$$

$$y_n = e^{-\Delta t/\tau} y_{n-1} + (1 - e^{-\Delta t/\tau}) x_{n-1}$$ $\Rightarrow$ Exact Solution!

23.3 Higher-Order System

Linear *Differential* eq. of order p -> linear *Difference* eq. of order p

$$G(s) = \frac{Y(s)}{X(s)} = \frac{K(\tau_a s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

by exact discretization

$$y_n + a_1 y_{n-1} + a_2 y_{n-2} = b_1 x_{n-1} + b_2 x_{n-2}$$

where $a_1 = -e^{-\Delta t/\tau_1} - e^{-\Delta t/\tau_2}$, $a_2 = e^{-\Delta t/\tau_1} e^{-\Delta t/\tau_2}$

$$b_1 = K\left(1 + \frac{\tau_a - \tau_1}{\tau_1 - \tau_2} e^{-\Delta t/\tau_1} + \frac{\tau_2 - \tau_a}{\tau_1 - \tau_2} e^{-\Delta t/\tau_2}\right)$$

$$b_2 = K\left(e^{-\Delta t\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)} + \frac{\tau_a - \tau_1}{\tau_1 - \tau_2} e^{-\Delta t/\tau_2} + \frac{\tau_2 - \tau_a}{\tau_1 - \tau_2} e^{-\Delta t/\tau_1}\right)$$

If $\tau_a = \tau_2$ and $K = 1$ → 1st order filter type

for steady state ⇒ $y_n = y_{n-1} = y_{n-2} = \bar{y}$, $x_{n-1} = x_{n-2} = \bar{x}$

$$\frac{\bar{y}}{\bar{x}} = \frac{b_1 + b_2}{1 + a_1 + a_2} \Rightarrow \text{gain!}$$

23.4 Fitting Discrete-Time Equations To Process Data

1. fit continuous model ---------> discrete time model (Convert)
   (Existing simplified technique)

2. fit discrete time model directly : Using optimization technique
   obj = $\sum (y_n - \hat{y}_n)^2$
   Levenburg-Marquadt method (in between Newton and Secant methods)

Chap. 24 Dynamic Response of Discrete-Time System

24.1. Z-Transform

Continuous Signal ---------> Sampled Signal
f(t) f*(t)

Impulse sampling : $f^*(t) = \sum_{n=0}^{\infty} f(n\Delta t)\delta(t - n\Delta t)$

(δ : impulse or Dirac delta function)

$$(f(n\Delta t) = \int_{n\Delta t^-}^{n\Delta t^+} f^*(t)\,dt)$$

$$\mathcal{L}[f^*(t)] = F^*(s) = \sum_{n=0}^{\infty} f(n\Delta t)e^{-n\Delta ts}$$

$e^{-n\Delta ts}$: real translation theorem for $\delta(t - n\Delta t)$

Let $z \triangleq e^{s\Delta t}$

$$\begin{aligned} F(z) &\triangleq Z[f^*(t)] = \sum_{n=0}^{\infty} f(n\Delta t)z^{-n} \\ &\triangleq Z[f(t)] \\ &= Z[f^*(t)] = \sum_{n=0}^{\infty} f_n z^{-n} \end{aligned}$$: Z-transform

⇒ Z-Transform is a special case of Laplace Transform
: Infinite series, but $F(z)$ can be written in closed form if $F(s)$ is a rational function

∗ Step function : $S(t)$; $f(n\Delta t)$ = 1 for all $n \geq 0$ ($f_0 = 1 \rightarrow f(0^+) = 1$)

$$F(z) = 1 + z^{-1} + z^{-2} + \ldots\ldots.$$

For $|z| > 1$, $F(z) = \dfrac{1}{1 - z^{-1}}$

▶ $|z| > 1 \leftrightarrow e^{s\Delta t} > 1 \Rightarrow s > 0$ to have finite values!

∗ Exponential function : $f(t) = Ce^{-at}$

$$F(z) = \sum_{n=0}^{\infty} Ce^{-an\Delta t}z^{-n}$$

if $|e^{-an\Delta t}z^{-1}| < 1$, $F(z) = \dfrac{C}{1 - e^{-a\Delta t}z^{-1}}$

→ $s > -a$

(if $a < 0$, s should be $s > -a > 0$ for $e^{-at}e^{-st}$ to have limit if $a<0$)

Even for $s < -a$, the expression will be valid for all s using analytic extension theorem.

Properties of the Z-Transform

1. Linearity

$$Z\,[a_1 f_1(t) + a_2 f_2(t)] = a_1 Z\,[f_1(t)] + a_2 Z\,[f_2(t)]$$

2. Real Translation Theorem

$Z\,[f(t - i\Delta t)] = z^{-i} F(z)$ provided that $f(t) = 0$ for t <0

pf)

$$\begin{aligned} Z\,[f(t - i\Delta t)] &= \sum_{n=0}^{\infty} f\,(n\Delta t - i\Delta t)\; z^{-n} && (Let\;\; j = n - i) \\ &= \sum_{j=-i}^{\infty} f(j\Delta t)\; z^{-j-i} && (f(j\Delta t) = 0 \;\; \text{for} \;\; j < 0) \\ &= z^{-i} \sum_{j=0}^{\infty} f(j\Delta t)\; z^{(-j)} = z^{-i}\, F\,(z) \end{aligned}$$

3. Complex Translation Theorem

$$Z[e^{-at} f(t)] = F(z e^{a\Delta t})$$

pf)

$$\begin{aligned} Z\,[e^{-at} f(t)] &= \sum_{n=0}^{\infty} e^{-an\Delta t} z^{-n} \\ &= \sum_{i=0}^{\infty} f(n\Delta t)\; (z e^{a\Delta t})^{-n} \\ &= F(z e^{a\Delta t}) \end{aligned}$$

4. Initial Value Theorem

$$\lim_{n \to 0} f(n\Delta t) = \lim_{z \to \infty} (1 - z^{-1}) F(z) \qquad \text{for} \quad |z| > 1$$

5. Final Value Theorem

$$\lim_{n \to \infty} f(n\Delta t) = \lim_{z \to 1} (1 - z^{-1}) F(z) \qquad \text{for} \quad |z| > 1 \;\; \text{provided} \;\; f(\infty\Delta t) \;\; \text{exists.}$$

pf)

$$\begin{aligned} \lim_{z \to 1}\; (1 - z^{-1}) F(z) &= \lim_{z \to 1} (1 - z^{-1}) \sum_{n=0}^{\infty} f(n\Delta t) z^{-n} \\ &= \lim_{z \to 1} [f(0) + \{f(\Delta t) - f(0)\} z^{-1} + \{f(2\Delta t) - f(\Delta t)\} z^{-2} + \ldots\}] \end{aligned}$$

$$= [f(0) + f(\Delta t) - f(0) + f(2\Delta t) - f(\Delta t) + \ldots + f(\infty\Delta t) - f((\infty - 1)\Delta t)]$$

$$= f(\infty\Delta t) = \lim_{n\to\infty} f(n\Delta t)$$

Ex 24.1 For $f(t) = t$, $F(z) = ?$

$$f(n\Delta t) = n\Delta t \qquad F(z) = \sum_{n=0}^{\infty} n\Delta t z^{-n} = \Delta t \sum_{n=0}^{\infty} n z^{-n}$$

Let $S(z) = \sum_{n=0}^{\infty} n z^{-n}$,

$$S(z) - z^{-1}S(z) = z^{-1} + z^{-2} + z^{-3} + \ldots\ldots = \frac{1}{1-z^{-1}} - 1$$

$$\therefore \; S(z) = \frac{1}{(1-z^{-1})^2} - \frac{1}{(1-z^{-1})} = \frac{z^{-1}}{(1-z^{-1})^2}$$

$$\Rightarrow \; F(z) = \Delta t S(z) = \frac{\Delta t \, z^{-1}}{(1-z^{-1})^2}$$

Ex 24.2 $Z(\cos bt) = ?$ $\quad Z(e^{-at}\cos bt) = ?$

Using Euler's identity, $\cos(nb\Delta t) = \frac{1}{2}(e^{jnb\Delta t} + e^{-jnb\Delta t})$ where $j = \sqrt{-1}$

$$F(z) = Z(\cos bt) = \sum_{n=0}^{\infty} (\cos nb\Delta t) z^{-n}$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} e^{jnb\Delta t} z^{-n} + \sum_{n=0}^{\infty} e^{-jnb\Delta t} z^{-n} \right)$$

(Z-transformation for exponential functions)

$$= \frac{1}{2}\left(\frac{1}{1 - e^{jb\Delta t} z^{-1}} + \frac{1}{1 - e^{-jb\Delta t} z^{-1}} \right)$$

$$= \frac{1}{2}\left(\frac{2 - 2\left(\frac{e^{jb\Delta t} + e^{-jb\Delta t}}{2}\right) z^{-1}}{1 - 2\left(\frac{e^{jb\Delta t} + e^{-jb\Delta t}}{2}\right) z^{-1} + z^{-2}} \right)$$

$$= \frac{1 - z^{-1}\cos b\Delta t}{1 - 2z^{-1}\cos b\Delta t + z^{-2}}$$

(Note : if $b = \frac{2\pi n}{\Delta t}$, then $\cos b\Delta t = 1$ for $t = \Delta t, 2\Delta t \ldots\ldots n\Delta t$

$\Rightarrow$ which results $\frac{1}{1-z^{-1}}$: step function

$\Rightarrow$ $f_n = 1$ for all n → aliasing)

Using complex translation,

$$Z(e^{-at}\cos bt) = F(z e^{a\Delta t})$$

$$= \frac{1 - z^{-1} e^{-a\Delta t} \cos b\Delta t}{1 - 2z^{-1} e^{-a\Delta t} \cos b\Delta t + z^{-2} e^{-2a\Delta t}}$$

6. Modified Z-transform

(Special version of Z-transform for fractional time delays)

$\Theta = (N+\sigma)\Delta t$, $0 < \sigma < 1$, N is a positive integer

$$Z[f(t-\Theta)] = \sum_{n=0}^{\infty} f(n\Delta t - N\Delta t - \sigma\Delta t) z^{-n}$$

Let m=1-σ and k = n-N-1. Then,

$$Z[f(t-\Theta)] = \sum_{k=-N-1}^{\infty} f(k\Delta t + m\Delta t) z^{-k-N-1}$$

since $f(n\Delta t) = 0$ for n <0 → lower limit : k = 0

$$Z[f(t-\Theta)] = z^{-N-1} \sum_{k=0}^{\infty} f(k\Delta t + m\Delta t) z^{-k}$$

$$\therefore \quad F(z, m) \triangleq z^{-N-1} \sum_{n=0}^{\infty} f(k\Delta t + m\Delta t) z^{-k}$$

* m : modified z-transform variable

⇒ **Ogata has table for modified z-transform.**
theorems for initial value, complex translation, etc are valid

⇒ **it requires information between samples**
-> Should know f(t)

Ex 24.3

$$\begin{aligned} F(z, m) &= Z_m(e^{-at}) \qquad (N=0) \\ &= z^{-1} \sum_{n=0}^{\infty} e^{-a(k+m)\Delta t} z^{-k} \\ &= e^{-am\Delta t} z^{-1} \sum_{n=0}^{\infty} e^{-ak\Delta t} z^{-k} \\ &= e^{-am\Delta t} z^{-1} \frac{1}{1-e^{-a\Delta t} z^{-1}} \end{aligned}$$

For m=0 (σ=1), $F(z, m) = \dfrac{z^{-1}}{1-e^{-a\Delta t} z^{-1}}$

→ One unit time delay of e^{-at}

Ex 24.4

$F(s) = \dfrac{1}{s(s+a)}$ $(a > 0)$; $\left\{ = \dfrac{1}{a}\left(\dfrac{1}{s} - \dfrac{1}{s+a}\right) \right\}$, $\lim_{n\to\infty} f(n\Delta t) = ?$

$$F(z) = \frac{1}{a}\left(\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-a\Delta t}z^{-1}}\right)$$

$$\begin{aligned}\lim_{n\to\infty} f(n\Delta t) &= \frac{1}{a}\lim_{z\to 1}(1-z^{-1})\left(\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-a\Delta t}z^{-1}}\right)] \\ &= \frac{1}{a}\lim_{z\to 1}\left[1 - \frac{z-1}{z-e^{-a\Delta t}}\right] = \frac{1}{a}\end{aligned}$$

24.2. Inversion of Z-Transform

$F(z)$ -----------> $f(t)$
Not unique

$F(z)$ -----------> $f^*(t)$ or $f^*(n\Delta t)$
Unique

$$f^*(t) = Z^{-1}[F(z)]$$

$$F(z) = \frac{r_1}{1-p_1 z^{-1}} = r_1(1 + p_1 z^{-1} + p_1^2 z^{-2} + \ldots + p_1^n z^{-n})$$

$$\Rightarrow f(n\Delta t) = f_n = r_1\,(p_1)^n$$

Methods : a. Partial fraction expansion
b. Long Division
c. Contour integration

a) Partial fraction expansion

suppose $F(z) = \dfrac{V_1(z)}{V_2(z)}$

V_1 : kth order polynomial in z^{-1} excluding delay z^{-N}

V_2 : mth order monic polynomial in z^{-1}

(No positive power of z in numerator including delay)

$$\begin{aligned}F(z) &= \frac{V_1(z)}{(1-p_1 z^{-1})(1-p_2 z^{-1})\ldots(1-p_m z^{-1})} \\ &= \frac{r_1}{(1-p_1 z^{-1})} + \frac{r_2}{(1-p_2 z^{-1})} + \ldots + \frac{r_m}{(1-p_m z^{-1})}\end{aligned}$$

Use of Heaviside's rule with $z = \frac{1}{p_i}$

$$\Rightarrow f(n\Delta t) = z^{-1}\left[\frac{r_1}{(1-p_1 z^{-1})}\right] + z^{-1}\left[\frac{r_2}{(1-p_2 z^{-1})}\right] + \ldots + z^{-1}\left[\frac{r_m}{(1-p_m z^{-1})}\right]$$

$$= r_1(p_1)^n + r_2(p_2)^n + \ldots\ldots + r_m(p_m)^n \qquad \left(\frac{r_i}{(1-p_i z^{-1})} \Rightarrow r_i p_i^n\right)$$

* For simple case of $f(n\Delta t) = r_1 {p_1}^n$

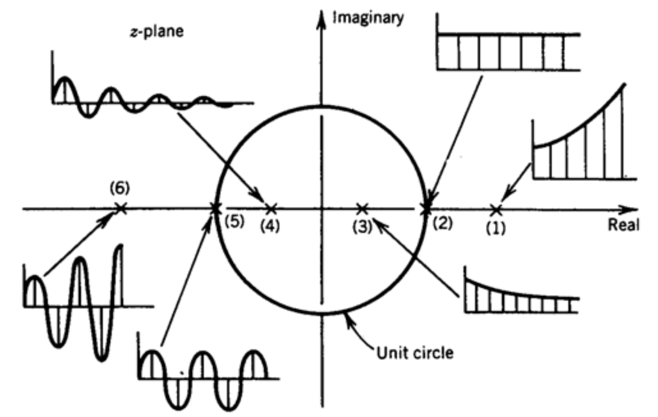
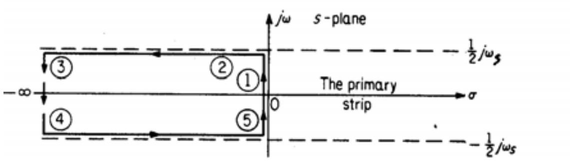
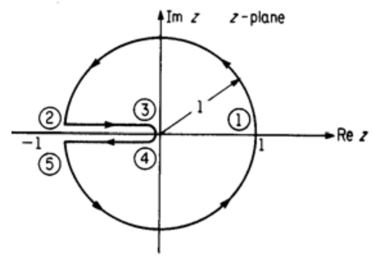


Figure 24.2 Time-domain responses for different locations of the root of $F(z)$.





Stable region:

① → ② $\quad s = j\omega \qquad 0 \le \omega \le \frac{1}{2}\omega_s$

$\Rightarrow z = e^{j\omega\Delta t} = 1.0\measuredangle\,\omega\Delta t, \quad 0 \le \omega \le \frac{1}{2}\omega_s$ (half cycle)

$\measuredangle 0 \le \measuredangle\,\omega\Delta t \le \measuredangle(\frac{1}{2}\omega_s\Delta t) \quad (\omega_s = 2\pi/\Delta t) \Rightarrow 0^\circ \le \omega\Delta t \le 180^\circ$ (Upper half cycle)

Imaginary axis → unit circle
1st and 4th quadrant → outside the unit circle

2nd and 3rd quadrant → inside the unit circle
point (0,0) → point (1,0)
negative real axis → segment (0,1)
positive real axis → segment (1,∞)
Re(s) = -∞ → point (0,0)
Im(s) = $\pm\frac{1}{2}j\omega_s$ with positive real→ segment (-1,-∞)
Im(s) = $\pm\frac{1}{2}j\omega_s$ with negative real → segment (-1,0)

- If $s=\sigma+j\omega$, $z=e^{(\sigma+j\omega)\Delta t}=e^{\sigma\Delta t}(\cos\omega\Delta t+j\sin\omega\Delta t)$
- For stability, $\sigma\le 0(\omega\ne 0)$ in s-domain for all ω, it spans the inside of the unit circle

- If $0\le p_i\le 1$, then

$$f(n\Delta t)=r_1 e^{-q_1 n\Delta t}+r_2 e^{-q_2 n\Delta t}+\ldots+r_m e^{-q_m n\Delta t}$$

where $q_i=-\frac{1}{\Delta t}\ln p_i$

- If $k>m$ (deg[$V_1(z)$] > deg[$V_2(z)$]), then partial fraction expansion is not strictly applicable

Ex. 24.5

$Z^{-1}[F(z)\}$ = ? when $F(z)=\dfrac{0.5z^{-1}}{(1-z^{-1})(1-0.5z^{-1})}$ with $\Delta t=1$

Sol)

$$F(z)=\frac{r_1}{1-z^{-1}}+\frac{r_2}{1-0.5z^{-1}}$$

$$r_1=(1-z^{-1})F(z)|_{z=1}=\frac{0.5}{0.5}=1$$

$$r_2=(1-0.5z^{-1})F(z)|_{z=0.5}=\frac{0.5*2}{1-2}=-1$$

$$=\frac{1}{1-z^{-1}}-\frac{1}{1-0.5z^{-1}}$$

where $q_1=-\frac{1}{\Delta t}\ln(1)=0$, $q_2=-\frac{1}{\Delta t}\ln(0.5)=0.693$

$$f(n\Delta t)=1-e^{-0.693n\Delta t}=1-(0.5)^n$$

b) Long Division

$$F(z) = \sum_{n=0}^{\infty} f(n\Delta t) z^{-n}$$
$$= f(0) + f(\Delta t) z^{-1} + f(2\Delta t) z^{-2} + \ldots\ldots\ldots$$

$$F(z) = \frac{V_1(z)}{V_2(z)} = C_0 + C_1 z^{-1} + C_2 z^{-2} + \ldots\ldots.$$

Ex 24.6 Solve for Ex24.5 using long division.

$$\begin{array}{r|l} & 0.5z^{-1} + 0.75z^{-2} + 0.875z^{-3} + 0.9375z^{-4} + 0.9687z^{-5} + \ldots \\ \hline 1 - 1.5z^{-1} + 0.5z^{-2} & 0.5z^{-1} \end{array}$$

→ Not in a closed form

c) Contour Integration

$$f(n\Delta t) = \frac{1}{2\pi j} \int_{\Gamma} F(z) z^{n-1} dz$$

where Γ should be appropriately specified.

cf) For inverse Laplace transform, $f(t) = \dfrac{1}{2\pi j} \lim_{\beta \to \infty} \int_{\alpha - j\beta}^{\alpha + j\beta} e^{st} F(s) ds$

- This method is seldom used in practice.

24.3. The Pulse Transfer Function

- Counterpart of Laplace domain "Transfer Function"

- input : $x(n\Delta t)$, $X(z)$
- output : $y(n\Delta t)$, $Y(z)$

- Convolution integral

$$y(t) = \int_0^t g(t-\tau) x^*(\tau) d\tau \qquad (g(t-\tau) : \text{impulse response})$$

where $x^*(\tau) = \sum_{k=0}^{\infty} x(k\Delta t)\delta(\tau - k\Delta t)$

$$y(t) = \int_0^t g(t-\tau) \sum_{k=0}^{\infty} x(k\Delta t)\delta(\tau - k\Delta t) d\tau$$

$$= \sum_{k=0}^{\infty} g(t - k\Delta t) x(k\Delta t)$$

for $t = n\Delta t$,

$$y(n\Delta t) = \sum_{k=0}^{\infty} g(n\Delta t - k\Delta t) x(k\Delta t)$$

$$\therefore \quad Y(z) = \sum_{n=0}^{\infty} y(n\Delta t) z^{-n} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} g(n\Delta t - k\Delta t) x(k\Delta t) z^{-n}$$

(Let $i = n - k$)

$$= \sum_{i=-k}^{\infty} \sum_{k=0}^{\infty} g(i\Delta t) z^{-(i+k)} x(k\Delta t)$$

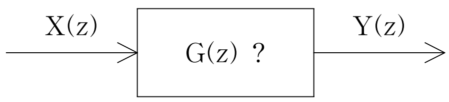
($g(i\Delta t) = 0$ for $i<0$)

$$= \sum_{i=0}^{\infty} g(i\Delta t) z^{-i} \sum_{k=0}^{\infty} x(k\Delta t) z^{-k}$$

$$= G(z) X(z)$$

$\therefore \quad G(z) \triangleq \sum_{i=0}^{\infty} g(i\Delta t) z^{-i}$: Pulse Transfer function

Ex. 24.7



if $G(s) = \dfrac{k}{\tau s + 1}$

Sol)

$$g(t) = \mathcal{L}^{-1}[G(s)] = \frac{k}{\tau} e^{-t/\tau}$$

$$G(z) = \frac{k}{\tau} \sum_{n=0}^{\infty} e^{-n\Delta t/\tau} z^{-n} = \frac{k/\tau}{1 - e^{-\Delta t/\tau} z^{-1}}$$

Ex. 24.8 Find the Step response.

$$G(z) = \frac{-0.3225 z^{-1} + 0.5712 z^{-2}}{1 - 0.9744 z^{-1} + 0.2231 z^{-2}} \qquad \left(= \frac{-0.3225 z^{-1}(1 - 1.771 z^{-1})}{(1 + a z^{-1})(1 + b z^{-1})} \right)$$

Sol) for unit step input $X(Z) = \frac{1}{1-z^{-1}}$,

$$Y(z) = G(z)X(z) = G(z)\frac{1}{1-z^{-1}}$$

by long division

$$= -0.3225z^{-1} - 0.0665z^{-2} + 0.2568z^{-3} + 0.5136z^{-4} + 0.6918z^{-5} + 0.8082z^{-6} + 0.8820z^{-7} + 0.9277z^{-8}$$

(First two coefficients have negative sign → inverse response)

Gain (z=1) = $\frac{-0.3225+0.5712}{1-0.9744+0.2231} = \frac{0.2487}{0.0256+0.2231} = 1$

24.4. Relating Pulse Transfer function to difference Equation

A general difference equation :

$$a_0 y_n + a_1 y_{n-1} + \ldots\ldots + a_m y_{n-m} = b_0 x_n + b_1 x_{n-1} + \ldots . b_k x_{n-k}$$

(From real translation theorem : $Z[y^{n-i}] = z^{-i}Y(z)$)

$$Y(z)(a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_m z^{-m}) = X(z)(b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_k z^{-k})$$

$$G(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} \ldots . b_k z^{-k}}{a_0 + a_1 z^{-1} + a_2 z^{-2} \ldots . a_m z^{-m}}$$

($b_0 \neq 0$ → immediate effect of input on output)

* Physical Realizability

From above eq'n, $a_0 \neq 0$ → physically realizable
(not depending on future inputs)

* The Zero-Order Hold (ZOH)

$$H(s) = \frac{1-e^{-s\Delta t}}{s}$$

Ex. 24.10

$G(s) = \frac{1}{\tau s + 1}$: difference equation model for ZOH plus first-order process

Sol)

$$H(s)G(s) = \frac{1 - e^{-s\Delta t}}{s} \frac{1}{\tau s + 1}$$
$$= \left(\frac{1}{s} - \frac{1}{s + 1/\tau}\right) - e^{-s\Delta t}\left(\frac{1}{s} - \frac{1}{s + 1/\tau}\right)$$

$$HG(z) = Z[H(s)G(s)] = Z\{\mathcal{L}^{-1}[H(s)G(s)]\}$$
$$= \left(\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-\Delta t/\tau} z^{-1}}\right) - z^{-1}\left(\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-\Delta t/\tau} z^{-1}}\right)$$
$$= \frac{z^{-1}(1 - e^{-\Delta t/\tau})}{1 - e^{-\Delta t/\tau} z^{-1}}$$

Let $a_1 \triangleq e^{-\Delta t/\tau}$, $\frac{Y(z)}{X(z)} = HG(z) = \frac{(1 - a_1) z^{-1}}{1 - a_1 z^{-1}}$

$$\Rightarrow \; y_n - a_1 y_{n-1} = (1 - a_1) x_{n-1}$$

* $HG(z) = Z[H(s)G(s)] = (1 - z^{-1}) \; Z\left[\frac{G(s)}{s}\right]$

$HG(z) \neq H(z)G(z)$

($\because$ $H(z) = Z\left[\frac{1 - e^{-s\Delta t}}{s}\right] = (1 - z^{-1}) Z\left[\frac{1}{s}\right] = \frac{1 - z^{-1}}{1 - z^{-1}} = 1$)

$(1 - z^{-1}) Z\left[\frac{G(s)}{s}\right] \neq Z\,[G(s)]$

Ex. Prove $\lim_{\Delta t \to 0} HG(z) = G(s)$ for first orfer T.F.

$$\lim_{\Delta t \to 0} HG(z) = \lim_{\Delta t \to 0} \frac{e^{-s\Delta t}(1 - e^{-\Delta t/\tau})}{1 - e^{-\Delta t/\tau} e^{-s\Delta t}} = \lim_{\Delta t \to 0} \frac{-se^{-s\Delta t} + (s + 1/\tau) e^{(s+1/\tau)\Delta t}}{(s + 1/\tau)\, e^{(s+1/\tau)\Delta t}}$$
$$= \frac{-s + s + 1/\tau}{s + 1/\tau} = \frac{1}{\tau s + 1} = G(s)$$

c.f.) $\lim_{\Delta t \to 0} G(z) = \lim_{\Delta t \to 0} \frac{1/\tau}{1 - e^{-\Delta t/\tau} e^{-s\Delta t}} = \infty \neq G(s)$

Ex. 24.11

$$G(s)=\frac{Y(s)}{X(s)}=\frac{1}{s} \text{ with ZOH } \rightarrow \text{ difference equation ?}$$

Sol)

$$HG(z)=Z[H(s)G(s)]=Z[(1-e^{-\Delta ts})/s^2]=\frac{\Delta tz^{-1}}{1-z^{-1}}$$

$$\left(Z[\frac{1}{s^2}]=\frac{\Delta tz^{-1}}{(1-z^{-1})^2} \right)$$

$\Rightarrow y_n - y_{n-1} = \Delta t\, x_{n-1}$

by long division, $y_n=\Delta t\sum_{k=1}^{n} x_{k-1}$ (integrating element)

*** High-Order System**

$$G(s)=\frac{Y(s)}{X(s)}=\frac{Ke^{-\Theta s}}{(\tau_1 s+1)(\tau_2 s+1)} \text{ with ZOH and time delay}(N\Delta t)$$

$$\Rightarrow G(z)=\frac{(b_1+b_2z^{-1})z^{-N-1}}{1+a_1z^{-1}+a_2z^{-2}}$$

$\Rightarrow$ apparent time delay is one sampling period longer.

24.5. Effect of Pole and Zero locations

- negative pole near unit circle has a pronounced effect
- ringing : alternation in sign of y
  (Poles are easy)

- But the zero location is unpredictable, mainly due to sampling effect
  $\Rightarrow$ No apparent simple relation

24.6. Conversion between Laplace and Z-Transform

- No ZOH is explicitly considered.
- Pade's approximation

$$z^{-1}=e^{-s\Delta t}\cong\frac{2-s\Delta t}{2+s\Delta t} \Rightarrow s\cong\frac{2}{\Delta t}\frac{1-z^{-1}}{1+z^{-1}}$$

$\Rightarrow$ Tustin's method (bilinear transformation)

- Power Series expansion

$$z^{-1} = e^{-s\Delta t} = 1 - s\Delta t + \frac{s^2 \Delta t^2}{2} - \ldots\ldots$$

$$\Rightarrow \; z^{-1} \cong 1 - s\Delta t \; \Rightarrow \; s \cong \frac{1 - z^{-1}}{\Delta t}$$ (Backward difference formula)

- Approximate Z-Transform

$$s = \frac{1 - z^{-1}}{\Delta t}$$

$$s^2 = z \left(\frac{1 - z^{-1}}{\Delta t} \right)^2$$

$$s^3 = \frac{2z}{(1 + z^{-1})} \left(\frac{1 - z^{-1}}{\Delta t} \right)^3$$

- Boxer-Thaler

$$s^2 = \frac{12}{(1 + 10z^{-1} + z^{-2})} \left(\frac{1 - z^{-1}}{\Delta t} \right)^2$$

$$s^3 = \frac{2z}{(1 + z^{-1})} \left(\frac{1 - z^{-1}}{\Delta t} \right)^3$$

Ex. 24.12

PID Controller : $G_c(s) = K_c(1 + \frac{1}{\tau_I s} + \tau_D s)$ → obtain velocity form digital PID

Sol) Using $s \cong \frac{1 - z^{-1}}{\Delta t}$

$$G_c(z) = \frac{K_c(a_0 + a_1 z^{-1} + a_2 z^{-2})}{1 - z^{-1}}$$

$$a_0 = 1 + \frac{\Delta t}{\tau_I} + \frac{\tau_D}{\Delta t}$$

$$a_1 = -(1 + \frac{2\tau_D}{\tau_I})$$

$$a_2 = \frac{\tau_D}{\Delta t}$$

let e_n : error, p_n : output from the controller

$$G_c(z) = \frac{P(z)}{E(z)}$$

$$(1-z^{-1})P(z) = K_c(a_0 + a_1 z^{-1} + a_2 z^{-2})E(z)$$

$$= p_n - p_{n-1} = \Delta p_n = K_c a_0 e_n + K_c a_1 e_{n-1} + K_c a_2 e_{n-2}$$

$$\therefore \quad \Delta p_n = K_c[\,(e_n - e_{n-1}) + \frac{\Delta t}{\tau_I} e_n + \frac{\tau_D}{\Delta t}(e_n - 2e_{n-1} + e_{n-2})]$$

* Tustin transformation will not give the same form.

Chap. 25 Analysis of Sampled Data Control System

- Block multiplication is different from Laplace transform case!

25.1 Open-Loop Block Diagram Analysis

- Synchronous sampling or process input / output

Figure 25.1 Block diagram with sampled input and output signals.

$$X^*(s) = \sum_{n=0}^{\infty} x(n\Delta t) e^{-n\Delta t s}$$

$$X(z) = \sum_{n=0}^{\infty} x(n\Delta t) z^{-n}$$

$$Y(s) = G(s)X^*(s)$$

$$Y^*(s) = [G(s)X^*(s)]^* = G^*(s)X^*(s)$$

[The Proof of this is in Franklin & Powell, p86, using freq.-domain analysis]

$$Y(z) = G(z)X(z)$$

* Pulse T.F. of Systems in Series

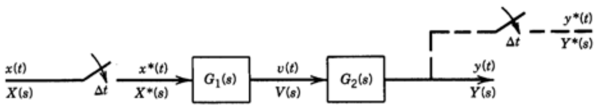


Figure 25.3 Two continuous systems in series with sampled input and output signals.

$$Y(s) = G_1(s)G_2(s)X^*(s)$$

$$Y^*(s) = [G_1(s)G_2(s)]^* X^*(s)$$

$$\frac{Y(z)}{X(z)} = Z[G_1(s)G_2(s)] = G_1G_2(z)$$

($G_1G_2(z) \neq G_1(z)G_2(z)$ in general)

- If a sampler is in between $G_1(s)$ and $G_2(s)$,

$$Y(s) = G_2(s)\, V^*(s)$$

$$V^*(s) = G_1^*(s)\; X_1^*(s)$$

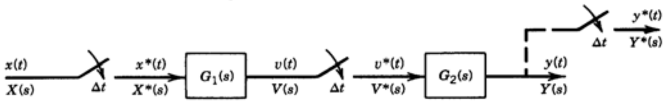


Figure 25.4 Two continuous systems in series with a sampler in-between.

$\Rightarrow\; Y(s) = G_2(s)\; G_1^*(s)\; X_1^*(s)$,

$\Rightarrow\; Y^*(s) = G_2^*(s)\; G_1^*(s)\; X_1^*(s)$, $\;\Rightarrow Y(z) = G_2(z)\, G_1(z)\, X(z)$

$$\frac{Y(z)}{X(z)} = G_2(z)\, G_1(z)$$

Ex 25.1 Show that $G_2(z)\,G_1(z) \neq G_2G_1(z)$ if $G_1(z)$ and $G_2(z)$, both first-order models.

pf) Let $G_1(s) = \dfrac{k_1}{\tau_1 s + 1}$ and $G_2(s) = \dfrac{k_2}{\tau_2 s + 1}$ $(\tau_1 \neq \tau_2)$,

and $a_1 = e^{-\Delta t/\tau_1}$, $a_2 = e^{-\Delta t/\tau_2}$,

$\Rightarrow\; G_1(z) = \dfrac{k_1/\tau_1}{(1 - a_1 z^{-1})}$, $\; G_2(z) = \dfrac{k_2/\tau_2}{(1 - a_2 z^{-1})}$

$$G_2(z)\, G_1(z) = \frac{k_1 k_2/\tau_1\tau_2}{(1 - a_1 z^{-1})(1 - a_2 z^{-1})}$$

also,

$$G_2(s)\, G_1(s) = \frac{k_1 k_2}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

From the Table 24.1,

$$G_2G_1(z) = Z[G_2(s)\, G_1(s)] = \frac{k_2 k_1 (a_2 - a_1) z^{-1}}{(\tau_2 - \tau_1)(1 - a_1 z^{-1})(1 - a_2 z^{-1})}$$

$\Rightarrow\; G_2(z)\, G_1(z) \neq G_2G_1(z)$

Ex. 25.2 $\;G_1(s) = H(s) = \dfrac{1 - e^{-s\Delta t}}{s}$ and $G_2(s) = \dfrac{K}{\tau s + 1}$

$x(t)$ is a unit step input

$\Rightarrow$ Examine the influence of the zero-order Hold. on $y(t)$

Sol) $V(s) = G_1(s) X^*(s) = \dfrac{1 - e^{-s\Delta t}}{s} \dfrac{1}{1 - e^{-s\Delta t}} = \dfrac{1}{s}$

$\therefore\; Y(s) = G_2(s)\; V(s) = \dfrac{k}{\tau s + 1} \dfrac{1}{s}$

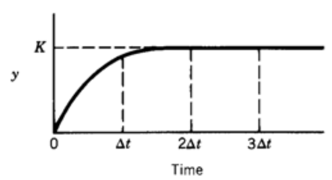
$Y(s)$ for $X^*(s)$ gives same results as $Y(s)$ for $X(s)$ of unit step input.

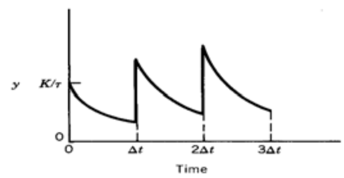
- For the case of sampling between $G_1(s)$ and $G_2(s)$

$$Y(s) = G_2(s)\ G_1^*(s) X^*(s) = \left(\frac{k}{\tau s+1}\right)\left(\frac{1-e^{-s\Delta t}}{s}\right)^* \left(\frac{1}{s}\right)^*$$

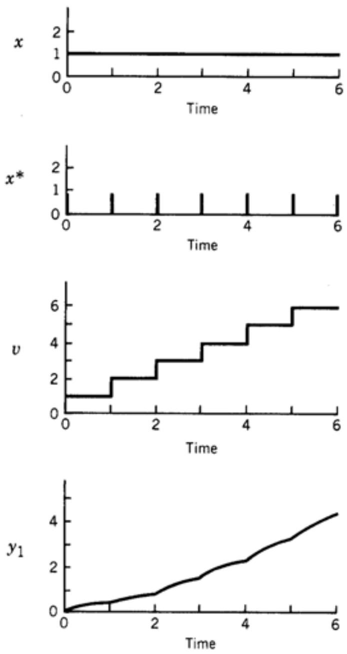
$$= \frac{k}{\tau s+1} \cdot 1 \cdot (1+e^{-s\Delta t}+e^{-2s\Delta t}+\ldots\ldots)$$

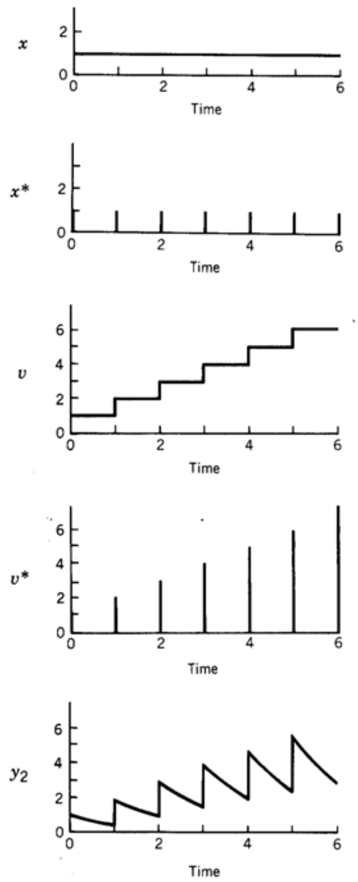
└→ (series of impulse input)





Ex. 25.3 $G_1(s)=\frac{1}{s}$, $G_2(s)=\frac{1}{2s+1}$, $X(s)=\frac{1}{s}$ with ideal sampler ⇒ $y(t)=$? for $0 \le t \le 6$





(The last graph for y_2 is wrong!)

$y_1^*(5) = 5.91$

$V(s) = G_1(s)X^*(s) = \frac{1}{s}\frac{1}{1-e^{-s\Delta t}}$

$Y(s) = G_2(s)V(s) = \frac{1}{s(2s+1)}\frac{1}{(1-e^{-s\Delta t})}$

$= \frac{1}{s(2s+1)}(1+e^{-s\Delta t}+e^{-2s\Delta t}+.....)$

$y_2^*(5) = 5.763$

$V(s) = G_1(s)X^*(s) = \frac{1}{(1-e^{-s\Delta t})^2}$

$Y(s) = G_2(s)V^*(s) = \frac{1}{2s+1}\frac{1}{(1-e^{-s\Delta t})^2}$

$= \frac{1}{(2s+1)}(1+e^{-s\Delta t}+e^{-2s\Delta t}+.....)^2$

$= \frac{1}{(2s+1)}(1+2e^{-s\Delta t}+3e^{-2s\Delta t}+4e^{-2s\Delta t}+...)$

* Pulse Transfer Functions of Systems in Parallel

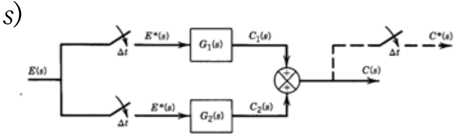
$C(s) = C_1(s) + C_2(s) = G_1(s)E^*(s) + G_2(s)E^*(s)$

$= [G_1(s) + G_2(s)]E^*(s)$

$\Rightarrow C^*(s) = [G_1^*(s) + G_2^*(s)]E^*(s)$

$\Rightarrow C^*(z) = [G_1(z) + G_2(z)]E^*(z)$

$\therefore \frac{C(z)}{E(z)} = G_1(z) + G_2(z)$

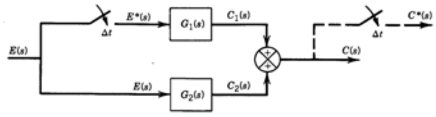


- Case of continuous load change

$C(s) = C_1(s) + C_2(s) = G_1(s)E^*(s) + G_2(s)E(s)$

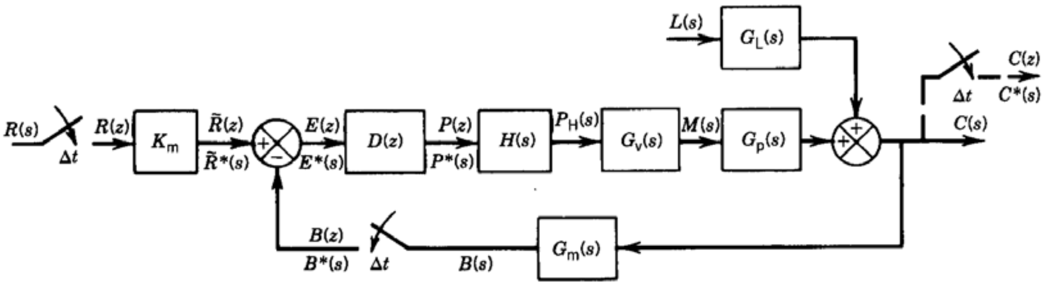
$C^*(s) = G_1^*(s)E^*(s) + [G_2(s)E(s)]^*$

$C(z) = G_1(z)E(z) + G_2E(z)$



$\Rightarrow$ **No pulse transfer function.**

25.2 Development of Closed-Loop Transfer Function



i) $L(s)=0$

$$B(s)=H(s)G_v(s)G_p(s)G_m(s)P^*(s)$$

$$\Rightarrow \quad B^*(s)=[HG_vG_pG_m]^*P^*(s)$$

$$P^*(s) = G_c^*(s)E^*(s)= G_c^*(s)[\widetilde{R}^*(s)-B^*(s)]$$

$$= G_c^*(s)[\widetilde{R}^*(s)-[HG_vG_pG_m]^*P^*(s)]$$

$$P^*(s)=\frac{G_c^*(s)\widetilde{R}(s)}{1+[HG_vG_pG_m]^*G_c^*(s)}$$

$$C(s)=H(s)G_v(s)G_p(s)P^*(s)$$

$$C^*(s)=[HG_vG_p]^*P^*(s)$$

$$C^*(s)=\frac{[HG_vG_p]^*G_c^*(s)\widetilde{R}^*(s)}{1+[HG_vG_pG_m]^*G_c^*(s)}$$

$$\widetilde{R}^*(s)=K_mR^*(s)$$

$$\frac{C^*(s)}{R^*(s)}=\frac{[HG_vG_p]^*G_c^*(s)K_m}{1+[HG_vG_pG_m]^*G_c^*(s)}$$

In Z-Transform notation,

$$\frac{C(z)}{R(z)}=\frac{K_mHG_vG_p(z)G_c(z)}{1+HG_vG_pG_m(z)G_c(z)}$$

The characteristic equation for the closed-loop control system,

$$1+HG_vG_pG_m(z)G_c(z)=0 \qquad \text{(determines stability)}$$

If $G_m(s)=K_m$,

$$\frac{C(z)}{R(z)}=\frac{HG(z)G_c(z)}{1+HG(z)G_c(z)}, \text{ where } HG(z)=HG_vG_pK_m(z)$$

ii) $R(s)=0$

For simplicity, let $G_m(s)=K_m$

$$\begin{cases} B(s)=H(s)G_v(s)G_p(s)K_mP^*(s)+G_L(s)K_mL(s) \\ B^*(s)=K_m[HG_vG_p]^*P^*(s)+K_m[G_LL]^* \end{cases} \qquad ①$$

and

$$\begin{cases} C(s) = H(s)G_v(s)G_p(s)P^*(s) + G_L(s)L(s) \\ C^*(s) = [HG_vG_p]^*P^*(s) + [G_LL]^* \end{cases} \quad ②$$

$$P^*(s) = -G_c^*B^*(s) \qquad (\because R(s) = 0) \quad ③$$

① and ② → ③ and solve for $P^*(s)$

$$C^*(s) = \frac{G_LL^*}{1 + K_mHG_vG_p^*(s)\ G_c^*(s)}$$

$$C(z) = \frac{G_LL(z)}{1 + K_m\ H\ G_v\ G_p(z)\ G_c(z)} = \frac{G_LL(z)}{1 + HG(z)\ G_c(z)}$$

- Because the disturbance $L(s)$ is not a sampled signal, closed-loop pulse T.F. cannot be found.

- But the characteristic equations for set-point changes and load change are same
  ⇒ Same stability analysis.

Ex. 25.4 For previous block diagram,

$$G_p(s) = \frac{K_pe^{-\Theta s}}{\tau_ps+1}, \quad G_L(s) = \frac{K_L}{\tau_Ls+1} \quad (\tau_p = \tau_L = \tau)$$

$$G_c(z) = K_c, \quad H(s) = \frac{1-e^{-s\Delta t}}{s}, \quad G_m = 1, \quad G_v = 1$$

$\Theta = N\Delta t$ and N is integer

⇒Find $C(z)$ for load change and characteristic equation? ($R(s) = 0$, $L(s) = \frac{1}{s}$)

Sol)

$$Z[G_L(s)L(s)] = Z[\frac{K_L}{\tau_Ls+1} \cdot \frac{1}{s}] = Z[\frac{K_L}{s} - \frac{K_L}{s+1/\tau}]$$

$$= K_L[\frac{1}{1-z^{-1}} - \frac{1}{1-az^{-1}}] = \frac{K_L(1-a)z^{-1}}{(1-z^{-1})(1-az^{-1})}$$

* Let $a = e^{-\Delta t/\tau}$

$$Z[H(s)G_p(s)] = Z[\frac{1-e^{-s\Delta t}}{s}\frac{K_p}{\tau_s+1}e^{-\Theta s}]$$

$$= Z[(e^{-Ns\Delta t} - e^{-(N+1)s\Delta t})\frac{1}{s} \cdot \frac{k_p}{\tau s+1}]$$

$$= K_p z^{-N}(1-z^{-1}) Z[\frac{1}{s} \cdot \frac{1}{\tau s+1}]$$

$$= K_p z^{-N}(1-z^{-1}) \frac{(1-a)z^{-1}}{(1-z^{-1})(1-az^{-1})} = K_p z^{-N-1} \frac{(1-a)}{(1-az^{-1})}$$

$$\therefore \quad C(z) = \frac{\dfrac{K_L(1-a)z^{-1}}{(1-z)(1-az^{-1})}}{1+\dfrac{z^{-N-1}K_cK_p(1-a)}{1-az^{-1}}} = \frac{K_L(1-a)z^{-1}}{(1-z^{-1})(1-az^{-1}+z^{-N-1}K_cK_p(1-a))}$$

$$= \frac{K_L(1-a)z^{-1}}{1-(1+a)z^{-1}+az^{-2}+K_cK_p(1-a)z^{-N-1}-K_cK_p(1-a)z^{-N-2}}$$

(Characteristic equation : the order of the equation depends on the time delay)

For Continuous System,

$$\frac{C(s)}{R(s)} = \frac{\dfrac{K_p e^{-\Theta s}}{\tau s+1} K_c}{1+\dfrac{K_p e^{-\Theta s}}{\tau s+1} \cdot K_c} = \frac{K_c K_p e^{-\Theta s}}{\tau s+1+K_p K_c e^{-\Theta s}}$$

Characteristic equation : $\tau s+1+K_cK_p e^{-\Theta s}=0$

$$j\tau\omega+1+K_cK_p e^{-j\Theta w} = j\tau w+1+K_cK_p(\cos\Theta\omega - j\sin\Theta\omega)=0$$

$$1+K_cK_p\cos\Theta\omega=0 \quad \text{and} \quad \tau\omega-K_cK_p\sin\Theta\omega=0$$

$$\cos\Theta\omega=-\frac{1}{K_cK_p} \quad \text{and} \quad \sin\Theta\omega=\frac{\tau\omega}{K_cK_p}$$

$\Rightarrow$ $\tan\Theta\omega=-\tau\omega$ $\Rightarrow$ infinite # of solution (periodic)

(while the discrete characteristic equation has limited # of roots.)

Ex 25.5 For $G_c=K_c$, $G_p=\dfrac{K_p}{\tau_p s+1}$, $G_m=G_v=1$

What is $\lim_{n\to\infty} C(n\Delta t)$ for $R(z)=\dfrac{1}{1-z^{-1}}$?

sol) $HG(z)=\dfrac{K_p(1-a)z^{-1}}{1-az^{-1}}$ $\quad (a=e^{-\Delta t/\tau_p})$

$$\frac{C(z)}{R(z)} = \frac{\dfrac{K_cK_p(1-a)z^{-1}}{1-az^{-1}}}{1+\dfrac{K_cK_p(1-a)z^{-1}}{1-az^{-1}}} = \frac{K_cK_p(1-a)z^{-1}}{1+[K_cK_p(1-a)-a]z^{-1}}$$

for the stable pole, $|z^{-1}|>1$

$|z|=|K_cK_p(1-a)-a|<1$

i) $K_cK_p(1-a)<1+a \Rightarrow K_cK_p<\dfrac{1+a}{1-a}$

ii) $K_cK_p(1-a)>-(1-a) \Rightarrow K_cK_p>-1$ (always satisfied due to negative feedback)

* $K_cK_p<\dfrac{1+a}{1-a}\rightarrow\infty$ as $a\rightarrow 1$ ($\Delta t\rightarrow 0$)

$\therefore$ if $\Delta t\rightarrow 0$, K_c can be infinite

(continuous system of 1st order + P-control does not cause stability problem for any K_c)

* $K_cK_p<\dfrac{1+a}{1-a}\rightarrow 1$ as $a\rightarrow 0$ ($\Delta t\rightarrow\infty$)

For stable processes with a unit step input,

$$\lim_{z\rightarrow 1}(1-z^{-1})C(z)=\frac{K_cK_p(1-a)}{1+K_cK_p(1-a)-a}=\frac{K_cK_p}{K_cK_p+1}$$

As $K_c\rightarrow\infty$, $C(z)\rightarrow 1$ which follows $R(z)$! Otherwise, shows offset. ($C(z)<R(z)$).

25.3 Stability of Sampled-Data Control System

* **Definition** : A linear sampled-data system is stable if the output sequence { $y(n\Delta t)$} is bounded for any bounded input sequence { $x(n\Delta t)$}. Otherwise, the system is said to be unstable. (BIBO stability)

$\Rightarrow$ No mention of the process response during the intersampled period (can be unstable for continuous system)

It can be detected by changing sampling period or by using modified z-transform.

Asymptotically stable
Weakly stable
Bounded-input Bounded-state (BIBS) stable : stronger than the BIBO stable
Exponentially stable
Marginally stable

* **The necessary and sufficient condition for stability** of a linear sampled-data system.

1) $\sum_{n=0}^{\infty}|g(n\Delta t)| < \infty$ (non-decaying response will be excluded!) or,

2) $G(z)$ has no poles on or outside the unit circle in z-plane.

* Stability Test

1. Modified Routh Stability Criterion

- The bilinear transformation $z = \frac{1+w}{1-w}$ is not exact (z-plane to s-plane), but the stability boundary is mapped exactly.

- Characteristic equation : $\Gamma(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots.. + a_1 z + a_0 = 0$

$$\Rightarrow \Gamma(w) = \bar{a}_n w^n + \bar{a}_{n-1} w^{n-1} + \ldots.. + \bar{a}_1 w + \bar{a}_0 = 0$$

($\bar{a}_i$ = real constant, $\bar{a}_i \neq a_i$, generally)

Stability condition : ($\bar{a}_n > 0$ w.l.o.g.)

i) $\bar{a}_0 \ldots.. \bar{a}_n$ are positive

ii) all elements in left column of Routh array are positive
⇒ # of sign change = # of unstable poles.

Ex 25.6

$$T(z) = 2z^3 + z^2 + z + 1 = 0$$

$$T(w) = 2\left(\frac{1+w}{1-w}\right)^3 + \left(\frac{1+w}{1-w}\right)^2 + \left(\frac{1+w}{1-w}\right) + 1 = 0$$

$$\Rightarrow w^3 + 7w^2 + 3w + 5 = 0$$

1	3
7	5
16/7	0
5	

⇒ Stable (all elements in the first column are positive)

2. Jury's Stability Criteria

- Apply directly to polynomial in z (No # of unstable poles)

Stability Condition ($a_n > 0$, a_i are real)

1) $\Gamma(z=1) > 0$
2) $\Gamma(z=-1) > 0$ for even n
   $\Gamma(z=-1) < 0$ for odd n
3) $|a_0| < a_n$
   $|b_0| > |b_{n-1}|$
   $|c_0| > |c_{n-2}|$ (n-1) Constraints
   $\vdots$
   $|s_0| > |s_2|$

<Jury Array>

$a_0 \; a_1 \; . \; . \; . \; . \; . \; a_n$
$a_n \; a_{n-1} \; . \; . \; . \; . \; . \; a_0$
$b_0 \; b_1 \; . \; . \; . \; b_{n-1}$
$b_{n-1} \; b_{n-2} \; . \; . \; . \; b_1$
$\vdots$
$r_0 \; r_1 \; r_2 \; r_3$
$r_3 \; r_2 \; r_1 \; r_0$
$s_0 \; s_1 \; s_2$
Total : (2n-3) rows

where $b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}$ $c_k = \begin{vmatrix} b_0 & b_{n-k} \\ b_n & b_k \end{vmatrix}$ $\cdots$ $s_k = \begin{vmatrix} r_0 & r_{3-k} \\ r_3 & r_k \end{vmatrix}$

Ex 25.7 $\Gamma(z) = 2z^4 - 3z^3 + 2z^2 - z + 1 = 0$

i) & ii) $\Gamma(1) = 1 > 0$, $\Gamma(-1) = 9 > 0$ (n = 4, even)

iii) Jury array

①	1	-1	2	-3	2	⇒ 1 < 2 (ok)
②	2	-3	2	-1	1	
③	-3	5	-2	-1		⇒ 3 > 1 (ok)
④	-1	-2	5	-3		
⑤	8	-17	11			⇒ 8 ≯ 11 (violated)

⇒ Unstable

- If either first or last element is zero ⇒ Use special technique

3. Schur-Cohn Criteria

More complicated (About twice as many determinants must be calculated)
Ref. : Ogata. "Discrete-Time Control System", 1987

*** Special Case of 0 element in the first column for Routh array**

- 0 means a pair of imaginary roots

- replace 0 to ε, and proceed the calculation

- if above 0 and below 0 have sign change ⇒ consider as one sign change

$s^3 + 2s^2 + s + 2 = 0$

| 1 1
| 2 2
| 0 (→ replace with ε)
| 2

$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$

| 1 24 -25
| 2 48 -50 (→use as aux. polynomial)
| 0 0
⇒ $P(s) = 2s^4 + 48s^2 - 50$

- if any derived row has all zero elements, use auxiliary polynomials,
⇒ two real roots with opposite sign radially and/or two conjugated imaginary roots

$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = (s^2 - 1)(s^2 + 25)(s + 2)$

| 1 24 -25
| 2 48 -50 ⇒ $P(s) = 2s^4 + 48s^2 - 50$
| 8 96 ⇐ replace 0-row with $dP/ds = 8s^3 + 96s$
| 24 -50
| 112.7 0
| -50

⇒ *4th* order auxiliary polynomial (2 pairs of radially symmetric roots)
+ 1 sign change (one root with positive real)

⇒ a pair of conjugate imaginary roots
+ a pair of radially symmetric roots (one root with positive real)

⇒ Deg (Aux. Polynomial)/2 = no. of pairs of radially symmetric roots
(The order of auxiliary polynomial will always be even!)

*** If $s = z - \sigma$ ($\sigma =$ constant)**

We can test the roots which lie to the right of the vertical line $s = -\sigma$